

Concurrence and a proper monogamy inequality for arbitrary quantum states

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We obtain an analytical lower bound of entanglement quantified by concurrence for arbitrary bipartite quantum states. It is shown that our bound is tight for some mixed states and is complementary to the previous known lower bounds. On the other hand, it is known that the entanglement monogamy inequality proposed by Coffman, Kundu, and Wootters is in general not true for higher dimensional quantum states. Inducing from the new lower bound of concurrence, we find a proper form of entanglement monogamy inequality for arbitrary quantum states.

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Quantum entanglement is considered to be the most nonclassical manifestations of quantum mechanics and plays an important role not only in quantum information sciences but also in condensed-matter physics [1, 2]. Due to the decoherence which is in general unavoidable for quantum system one has to deal with mixed states in quantum information processing. However, it turns out that the detection, quantification and distillability of quantum entanglement for mixed states are much more complicated than expected though much progress has already been made in the past years. For example, the operational measure of entanglement for arbitrary mixed states is still not known, and even the separability criterion for mixed states can detect all entanglement for only $2 \otimes 2$ and $2 \otimes 3$ systems [3, 4].

The concurrence is one of the well accepted entanglement measure [5], however, the analytical formulae of concurrence are only for two-qubit states [5] and some high dimensional bipartite states with certain symmetries, like isotropic ones [6]. For general higher dimensional mixed states less has been known yet [7, 8] and the optimization method is necessary and is thus not operational. Instead of analytical exact results, some operational lower bounds of the concurrence have been derived recently [9, 10], which can detect some bound entangled states but not all. It is clear that the lower bound of concurrence can also provide a separability criterion.

On the other hand, the study of distributed entanglement for multipartite states is important for quantum cryptography [11] and condensed-matter physics [12]. Nevertheless the monogamy inequality developed by Coffman, Kundu, and Wootters holds only for qubit systems [11, 13], it does not hold in general for higher dimension if we use a straightforward extension [14]. It is generally accepted that the entanglement can not be shared freely, thus a proper definition of the general monogamy inequality of entanglement is necessary.

In this Letter, the situation above is shown to be dramatically improved: We derive an analytical lower bound of concurrence for arbitrary bipartite quantum states by decomposing the joint Hilbert space into many $2 \otimes 2$ di-

mensional subspaces, which does not involve any optimization procedure [8] and gives an effective evaluation of entanglement together with an operational sufficient condition for the distillability of any bipartite quantum states, which improves the result [15]. Inducing from our bound, for the first time, we generalize the monogamy inequality developed by Coffman, Kundu, and Wootters to any pure multipartite quantum states. This proper inequality not only is the fundamental constraint for entanglement sharing but also can define a measure of multipartite entanglement.

For a pure bipartite state $\rho_{AB} = |\psi\rangle\langle\psi|$ in a finite $d_1 \otimes d_2$ dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the concurrence is defined as $\mathcal{C}(|\psi\rangle) = \sqrt{2(1 - \text{Tr}\rho_A^2)}$ with $\rho_A = \text{Tr}_B \rho_{AB}$ the reduced density matrix. A pure state can be generally expressed as $|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \phi_{ij} |ij\rangle$, $\phi_{ij} \in \mathbb{C}$, in computational basis $|i\rangle$ and $|j\rangle$ of \mathcal{H}_A and \mathcal{H}_B respectively, $i = 1, \dots, d_1$ and $j = 1, \dots, d_2$. After some algebraic calculations [16], one derives the squared concurrence:

$$\mathcal{C}^2(|\psi\rangle) = \sum_{m=1}^{D_1} \sum_{n=1}^{D_2} |\mathcal{C}_{mn}|^2 = 4 \sum_{i < j}^{d_1} \sum_{k < l}^{d_2} |\phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk}|^2, \quad (1)$$

where $D_1 = d_1(d_1 - 1)/2$, $D_2 = d_2(d_2 - 1)/2$, $\mathcal{C}_{mn} = \langle\psi|\tilde{\psi}_{mn}\rangle$, $|\tilde{\psi}_{mn}\rangle = (L_m \otimes L_n)|\psi^*\rangle$, and $L_m, m = 1, \dots, d_1(d_1 - 1)/2$, $L_n, n = 1, \dots, d_2(d_2 - 1)/2$ are the generators of group $SO(d_1)$ and $SO(d_2)$ respectively.

From Eq.(1) it is evident that the $d_1 \otimes d_2$ dimensional Hilbert space is decomposed into $d_1(d_1 - 1)d_2(d_2 - 1)/4$ $2 \otimes 2$ dimensional subspaces, such that the squared concurrence is just the sum of all squared “two-qubit” concurrences. A pure state is separable iff all these “two qubits” are separable. For a mixed state ρ

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1, \quad (2)$$

the concurrence is defined by the convex-roof:

$$\mathcal{C}(\rho) \equiv \min \sum_i p_i \mathcal{C}(|\psi_i\rangle) \quad (3)$$

of all possible decompositions into the pure states $|\psi_i\rangle$. Although the concurrence (3) is cumbersome to solve due to a high dimensional optimization, one may provide an analytical lower bound on it as shown in the following.

Theorem 1: For an arbitrary $d_1 \otimes d_2$ state (2), the concurrence $\mathcal{C}(\rho)$ satisfies

$$\tau(\rho) \equiv \sum_{m=1}^{d_1(d_1-1)/2} \sum_{n=1}^{d_2(d_2-1)/2} \mathcal{C}_{mn}^2 \leq \mathcal{C}^2(\rho), \quad (4)$$

where τ is a lower bound of squared concurrence and

$$\mathcal{C}_{mn} = \max \{0, \lambda_{mn}^{(1)} - \lambda_{mn}^{(2)} - \lambda_{mn}^{(3)} - \lambda_{mn}^{(4)}\}, \quad (5)$$

with $\lambda_{mn}^{(1)}, \dots, \lambda_{mn}^{(4)}$ being the square roots of the four nonzero eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho \tilde{\rho}_{mn}$ where $\tilde{\rho}_{mn} = (L_m \otimes L_n) \rho^* (L_m \otimes L_n)$.

Proof: Set $|\xi_i\rangle = \sqrt{p_i} |\psi_i\rangle$. The concurrence (3) takes the form:

$$\mathcal{C}(\rho) = \min_i \sum_i \left(\sum_{m=1}^{d_1(d_1-1)/2} \sum_{n=1}^{d_2(d_2-1)/2} |\langle \xi_i | L_m \otimes L_n | \xi_i^* \rangle|^2 \right)^{\frac{1}{2}}. \quad (6)$$

For simplicity, we denote the term after the min as \mathcal{D} , i.e., $\mathcal{C}(\rho) = \min \mathcal{D}$. Recall that for any function $F = \sum_i \left(\sum_j x_{ij}^2 \right)^{1/2}$ subjected to the constraints $z_j = \sum_i x_{ij}$ with x_{ij} being real and nonnegative, the inequality $\sum_j z_j^2 \leq F^2$ holds, from which it follows that \mathcal{D} satisfies

$$\sum_{m=1}^{d_1(d_1-1)/2} \sum_{n=1}^{d_2(d_2-1)/2} \left(\sum_i |\langle \xi_i | L_m \otimes L_n | \xi_i^* \rangle| \right)^2 \leq \mathcal{D}^2. \quad (7)$$

In order to seek a lower bound of the minimum of \mathcal{D} over all pure-state decompositions, we only need to consider

$$\mathcal{C}'_{mn} = \min_i |\langle \xi_i | L_m \otimes L_n | \xi_i^* \rangle|, \quad (8)$$

for all m and n , by using procedure of extremizations adopted in [5, 8]. Let λ_i and $|\chi_i\rangle$ be eigenvalues and eigenvectors of ρ respectively. Any decomposition of ρ can be obtained from a unitary $d \times d$ matrix V_{ij} , $|\xi_j\rangle = \sum_{i=1}^d V_{ij}^* (\sqrt{\lambda_i} |\chi_i\rangle)$. Therefore one has $\langle \xi_i | L_m \otimes L_n | \xi_j^* \rangle = (V Y V^T)_{ij}$, where the matrix Y is defined by $Y_{ij} = \langle \chi_i | L_m \otimes L_n | \chi_j^* \rangle$. Eq.(8) turns out to be $\mathcal{C}'_{mn} = \min_i |[V Y V^T]_{ii}| = \lambda_{mn}^{(1)} - \sum_{i>1} \lambda_{mn}^{(i)}$ [5, 17], where $\lambda_{mn}^{(j)}$ are the square roots of the eigenvalues of the positive Hermitian matrix $Y Y^\dagger$, or equivalently the non-Hermitian matrix $\rho \tilde{\rho}_{mn}$ in decreasing order. As the matrix $L_m \otimes L_n$ has $d_1 d_2 - 4$ rows and $d_1 d_2 - 4$ columns that are identically zero, the matrix $\rho \tilde{\rho}_{mn}$ has a rank no greater than 4, i.e., $\lambda_{mn}^{(j)} = 0$ for $j \geq 5$. From the above analysis we have Eqs.(4)-(5). \square

Remark: Our bound τ (4) in fact characterizes some “two-qubit” entanglement in a high dimensional bipartite state. One can directly verify that there are at most $4 \times 4 = 16$ nonzero elements in each matrix $\tilde{\rho}_{mn}$ so as to lead to a 4×4 matrix $\varrho(\sigma_y \otimes \sigma_y) \varrho^*(\sigma_y \otimes \sigma_y)$, where σ_y is the Pauli matrix and the matrix ϱ is a submatrix of the original ρ :

$$\varrho = \begin{pmatrix} \rho_{ik,ik} & \rho_{ik,il} & \rho_{ik,jk} & \rho_{ik,jl} \\ \rho_{il,ik} & \rho_{il,il} & \rho_{il,jk} & \rho_{il,jl} \\ \rho_{jk,ik} & \rho_{jk,il} & \rho_{jk,jk} & \rho_{jk,jl} \\ \rho_{jl,ik} & \rho_{jl,il} & \rho_{jl,jk} & \rho_{jl,jl} \end{pmatrix}, \quad (9)$$

$i \neq j$ and $k \neq l$, with subindices i and j associated with the space \mathcal{H}_A , and k and l with the space \mathcal{H}_B . The “two-qubit” submatrix ϱ is not normalized but positive semidefinite, such that \mathcal{C}_{mn} is just the concurrence of the state (9).

Our bound τ also provides a much clearer structure of entanglement, which not only yields an effective separability criterion and an easy evaluation of entanglement, but also helps one to classify mixed-state entanglement. Based on the positive partial transpose (PPT) criterion, a necessary and sufficient condition for the distillability was proposed in [15], which is not operational in general. In the following, we derive an alternative distillability criterion based on our bound τ to improve the operationality to some degree.

Theorem 2: For any bipartite quantum state ρ , if $\tau(\rho^{\otimes N}) > 0$ for a certain positive integer N , ρ is distillable.

Proof: It was shown in [15] that a density matrix ρ is distillable iff there are some projectors A, B that map high dimensional spaces to two-dimensional ones and certain number N such that the state $A \otimes B \rho^{\otimes N} A \otimes B$ is entangled. Thus if $\tau(\rho^{\otimes N}) > 0$, there exists one submatrix of matrix $\rho^{\otimes N}$ similar as (9) which has non-zero τ and is entangled in $2 \otimes 2$ space. So we know that ρ is distillable. We remark that this submatrix which has a positive τ is the entangled state $A \otimes B \rho^{\otimes N} A \otimes B$ up to normalization. \square

Corollary 1: The lower bound $\tau(\rho) > 0$ is a sufficient condition for the distillability of any bipartite state ρ .

Corollary 2: The lower bound $\tau(\rho) = 0$ is a necessary condition for the separability of any bipartite state ρ .

Remark: Corollary 1 directly follows from Theorem 2 and this case is referred as 1-distillable [19]. The opposite direction of Theorem 2, whether $\tau(\rho^{\otimes N}) > 0$ is a necessary condition of distillability is still a challenging question. The answer to this question can not only provide an operational criterion for mixed states distillability but also shed light on question of whether the non-PPT (NPPT) nondistillable states exist which is studied numerically in [18, 19].

Our bound τ , PPT criterion, separability and distillability for any bipartite quantum state ρ have the following relations. If $\tau(\rho) > 0$, ρ is entangled. If ρ is separable, it

is PPT. If $\tau(\rho) > 0$, ρ is distillable. If ρ is distillable, it is NPPT. From the last two propositions it follows that if ρ is PPT, $\tau(\rho) = 0$, i.e., if $\tau(\rho) > 0$, ρ is NPPT. We give some examples below.

Example 1: Horodecki's $3 \otimes 3$ system [21]:

$$\sigma_\alpha = \frac{2}{7}|\Psi^+\rangle\langle\Psi^+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, \quad (10)$$

where $\sigma_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$, $\sigma_- = \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|)$, and $|\Psi^+\rangle$ is a maximally entangled state. The state σ_α is separable for $2 \leq \alpha \leq 3$; bound entangled for $3 < \alpha \leq 4$; free entangled for $4 < \alpha \leq 5$ [21]. From lower bound τ in (4) we have $\tau(\sigma_\alpha) = 0$ for $2 \leq \alpha \leq 4$ and $\tau(\sigma_\alpha) = 4 \left(2 - \sqrt{\alpha(5-\alpha)}\right)^2 / 147$ for $4 < \alpha \leq 5$. According to Corollary 1 since $\tau(\sigma_\alpha) > 0$ for $4 < \alpha \leq 5$, the state is distillable, agreeing with the conclusion in [21]. Note that our lower bound is weaker than the one by realignment [10] for $4 \leq \alpha \lesssim 4.79$ but stronger for $4.79 \lesssim \alpha \leq 5$.

Example 2: Isotropic states in $d \otimes d$ dimensions [6, 22]:

$$\rho_F = \frac{1-F}{d^2-1} (I - |\Phi^+\rangle\langle\Phi^+|) + F|\Phi^+\rangle\langle\Phi^+|, \quad (11)$$

where $|\Phi^+\rangle$ is a maximally entangled state. These states are separable for $F \leq 1/d$ [22]. Our bound τ gives $\tau(\rho_F) = 0$ for $F \leq 1/d$, and $\tau(\rho_F) = 2(dF-1)^2/d(d-1)$ for $F > 1/d$, which is just the exact squared concurrence [23]. Thus these states saturate the inequality (4), which implies that the entanglement of these states is composed of only the entanglement of the “two qubits” in each state. According to Corollary 1 since $\tau(\rho_F) > 0$ for $F > 1/d$, all these states are distillable, agreeing with the analysis in [15].

Now we show a NPPT quantum state with $\tau = 0$.

Example 3: Werner states in $3 \otimes 3$ dimensions [24]:

$$\rho_W(\lambda) = \frac{1}{8\lambda-1} \left(\lambda I - \frac{\lambda+1}{3} H \right), \quad (12)$$

where $H|i, j\rangle = |j, i\rangle$ for all $i, j = 1, 2, 3$. For any finite $\lambda > 0$ the state $\rho_W(\lambda)$ is a NPPT state [20]. It is conjectured that for $\lambda \geq 2$ the state $\rho_W(\lambda)$ is undistillable [18, 19]. Our lower bound τ of the state (12) $\tau(\rho_W(\lambda)) = (4-2\lambda)^2/3(8\lambda-1)^2 > 0$ for $0 < \lambda < 2$. Hence these states in this parameter region are distillable according to Corollary 1, agreeing with the analysis in [19]. While the lower bound is $\tau(\rho_W(\lambda)) = 0$ for $\lambda \geq 2$, these states are just the NPPT states with $\tau = 0$ and they are 1-copy undistillable. However, the nondistillability of N -copy does not imply the undistillability of $N+1$ -copy [25]. And the technique in Theorem 2 may help to confirm the long-standing conjecture by computing $\tau(\rho^{\otimes N})$.

From these examples one can explicitly see that our bound τ provides an easy evaluation of concurrence for most of the free entangled states. On the other hand, as

we known, the entanglement is monogamous [11], however, surprisingly, a direct extension of the monogamy inequality from qubit case to the general case does not work [1, 14]. In this Letter, interestingly, one fundamental property of our bound τ is shown that it is monogamous also. It is thus a proper definition of the monogamy inequality for general cases.

Theorem 3: For any pure tripartite state $|\psi\rangle_{ABC}$ in arbitrary $d_1 \otimes d_2 \otimes d_3$ dimensional spaces, the lower bound τ of concurrence satisfies

$$\tau(\rho_{AB}) + \tau(\rho_{AC}) \leq \tau(\rho_{A:BC}), \quad (13)$$

where $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|)$, $\rho_{AC} = \text{Tr}_B(|\psi\rangle_{ABC}\langle\psi|)$, and $\rho_{A:BC} = \text{Tr}_{BC}(|\psi\rangle_{ABC}\langle\psi|)$.

Proof: Since $\mathcal{C}_{mn}^2 \leq \left(\lambda_{mn}^{(1)}\right)^2 \leq \sum_{i=1}^4 \left(\lambda_{mn}^{(i)}\right)^2 = \text{Tr}(\rho\tilde{\rho}_{mn})$ for each m and n in Eqs.(4) and (5), one can derive the inequality:

$$\begin{aligned} \tau(\rho_{AB}) + \tau(\rho_{AC}) &\leq \sum_l^{D_1} \sum_k^{D_2} \text{Tr}[\rho_{AB}(\tilde{\rho}_{AB})_{lk}] \\ &+ \sum_p^{D_1} \sum_q^{D_3} \text{Tr}[\rho_{AC}(\tilde{\rho}_{AC})_{pq}], \quad (14) \end{aligned}$$

where $D_3 = d_3(d_3-1)/2$. By using a similar analysis in [11] one has $\sum_{lk} \text{Tr}[\rho_{AB}(\tilde{\rho}_{AB})_{lk}] = 1 - \text{Tr}\rho_A^2 - \text{Tr}\rho_B^2 + \text{Tr}\rho_C^2$ and $\sum_{pq} \text{Tr}[\rho_{AC}(\tilde{\rho}_{AC})_{pq}] = 1 - \text{Tr}\rho_A^2 + \text{Tr}\rho_B^2 - \text{Tr}\rho_C^2$, where $l, p = 1, \dots, D_1$, $k = 1, \dots, D_2$, $q = 1, \dots, D_3$. The sum of these two inequalities results in that the right-hand side of (15) equals to $2(1 - \text{Tr}\rho_A^2) = \mathcal{C}^2(\rho_{A:BC})$. Taking into account that $\tau(\rho_{A:BC}) = \mathcal{C}^2(\rho_{A:BC})$ for a pure state, one obtains the inequality (13). \square

Corollary 3: Subsequently, we have the general monogamy inequality

$$\tau(\rho_{AB_1}) + \tau(\rho_{AB_2}) + \dots + \tau(\rho_{AB_n}) \leq \tau(\rho_{A:B_1\dots B_n}), \quad (15)$$

for any pure multipartite quantum state $\rho_{AB_1B_2\dots B_n}$ and A, B_1, \dots, B_n may contain any number of particles, respectively.

To see the tightness of the inequality (13), we consider the following examples.

Example 4: Aharonov state of three qutrits:

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}(|012\rangle + |120\rangle + |201\rangle - |021\rangle - |102\rangle - |210\rangle).$$

For this state, it was shown that the original Coffman-Kundu-Wootters inequality is violated since $\mathcal{C}^2(\rho_{AB}) + \mathcal{C}^2(\rho_{AC}) = 2 > \mathcal{C}^2(\rho_{A:BC}) = \frac{4}{3}$ [14]. According to Theorem 3 we have $\tau(\rho_{AB}) + \tau(\rho_{AC}) = \frac{2}{3} < \tau(\rho_{A:BC}) = \frac{4}{3}$, thus the inequality (13) is satisfied.

Example 5: Generalized five-qubit W state:

$$\begin{aligned} |\phi\rangle_{ABC} &= \alpha|10000\rangle + \beta|01000\rangle + \gamma|00100\rangle \\ &+ \delta|00010\rangle + \eta|00001\rangle, \quad (16) \end{aligned}$$

where the subsystem B (resp. C) contains the second and third (resp. the last two) qubits. The ABC system is $2 \otimes 4 \otimes 4$ dimensional. For this state, one finds that $\tau(\rho_{AB}) = 4|\alpha|^2(|\beta|^2 + |\gamma|^2)$, $\tau(\rho_{AC}) = 4|\alpha|^2(|\delta|^2 + |\eta|^2)$, and $\tau(\rho_{A:BC}) = 4|\alpha|^2(|\beta|^2 + |\gamma|^2 + |\delta|^2 + |\eta|^2)$. Thus $\tau(\rho_{AB}) + \tau(\rho_{AC}) = \tau(\rho_{A:BC})$, i.e., the state (16) saturates the inequality (13).

Due to the monogamy inequality, the difference between the two sides of (13) can be interpreted as a residual entanglement τ_{ABC} as for the qubit case [11]:

$$\tau_{ABC} = \tau(\rho_{A:BC}) - \tau(\rho_{AB}) - \tau(\rho_{AC}), \quad (17)$$

which, as usual, can be served as a measure of multipartite entanglement and is fully analytical. Moreover a pure tripartite state $|\psi\rangle_{ABC}$ can be expressed in the standard basis $\{|ijk\rangle\}$, where $i = 1, \dots, d_1, j = 1, \dots, d_2$, and $k = 1, \dots, d_3$: $|\psi\rangle_{ABC} = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \sum_{k=1}^{d_3} \phi_{ijk} |ijk\rangle$. Now the whole joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ can be decomposed into $d_1(d_1-1)d_2(d_2-1)d_3(d_3-1)/8 \otimes 2 \otimes 2 \otimes 2$ dimensional subspaces. While each $2 \otimes 2 \otimes 2$ dimensional subspace has a same form of the residual entanglement as the one [11] in terms of the coefficients ϕ_{ijk} . Therefore the residual entanglement (17) for any pure tripartite state takes the expression:

$$\tau_{ABC} = 4 \sum_{i,i'=1}^{d_1} \sum_{j,j'=1}^{d_2} \sum_{k,k'=1}^{d_3} |d_{ijk}^{(1)} - 2d_{ijk}^{(2)} + 4d_{ijk}^{(3)}|, \quad (18)$$

where

$$\begin{aligned} d_{ijk}^{(1)} &= \phi_{ijk}^2 \phi_{i'j'k'}^2 + \phi_{ijk}^2 \phi_{i'j'k}^2 + \phi_{ijk}^2 \phi_{i'jk'}^2 + \phi_{ijk}^2 \phi_{i'jk}^2, \\ d_{ijk}^{(2)} &= \phi_{ijk} \phi_{i'j'k'} \phi_{ij'k} \phi_{i'jk} + \phi_{ijk} \phi_{i'j'k'} \phi_{ij'k} \phi_{i'jk} \\ &\quad + \phi_{ijk} \phi_{i'j'k'} \phi_{ij'k} \phi_{i'jk} + \phi_{ijk} \phi_{i'j'k'} \phi_{ij'k} \phi_{i'jk}, \\ d_{ijk}^{(3)} &= \phi_{ijk} \phi_{i'j'k} \phi_{ij'k} \phi_{i'jk} + \phi_{ijk} \phi_{i'j'k} \phi_{ij'k} \phi_{i'jk}, \end{aligned} \quad (19)$$

with the constraints of the subindices $i < i', j < j'$ and $k < k'$. Since each $d_{ijk}^{(l)}$ is symmetrical with respect to i, j and k , it is invariant under permutations of the subsystems A, B and C . Thus the residual entanglement τ_{ABC} (18) is invariant under such permutations.

In summary, we have shown a novel lower bound of concurrence for any bipartite quantum states, which can be analytically obtained by calculating all “two-qubit” concurrences and is complementary to known results. Our bound becomes exact for some mixed states. It is an operational sufficient criterion for the distillability. With the form of the lower bound, the monogamy inequality developed by Coffman, Kundu, and Wootters is generalized to any pure multipartite quantum states. Consequently one can define a measure of multipartite entanglement, which can find wide potential applications in studying quantum phase transition [26, 27] and in seeking the ground-state energy of condensed-matter systems [28]. The method developed might also help to calculate

the entanglement of formation and the distillation rate of entanglement.

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